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Special loop connections of linear systems ☆

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Abstract

In this paper we completely determine the possible eigenvalues of a matrix of a system obtained as a result of special loop connections of arbitrary many linear systems.

Keywords: Linear systems; Loop connections; Eigenvalues; Matrix completions

1. Introduction

If $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and $i \in \{1, \ldots, t\}$, suppose that S_i is a continuous-time finite-dimensional linear time-invariant system described by the following system of ordinary differential equations of the first degree:

$$S_i \begin{cases} \dot{x}_i = A_i x_i + B_i u_i, \\ y_i = C_i x_i, \end{cases} \tag{1}$$

where $A_i \in \mathbb{K}^{n_i \times n_i}$, $B_i \in \mathbb{K}^{n_i \times m_i}$, $C_i \in \mathbb{K}^{p_i \times n_i}$, while x_i denotes the state, u_i denotes the input and y_i denotes the output of the system S_i, for details see [10].

If $\mathbb{F}$ is an arbitrary field and $i \in \{1, \ldots, t\}$, suppose that S_i is a discrete-time finite-dimensional linear time-invariant system described by the following system of equations:

$$S_i \begin{cases} x_{n+1}^i = A_i x_n^i + B_i u_n^i, \\ y_n^i = C_i x_n^i, \end{cases} \qquad n \in \mathbb{N}, \tag{2}$$

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where $A_i \in \mathbb{F}^{n_i \times n_i}$, $B_i \in \mathbb{F}^{n_i \times m_i}$ and $C_i \in \mathbb{F}^{p_i \times n_i}$, while x_n^i denotes the state, u_n^i denotes the input and y_n^i denotes the output of the system S_i, for details see [10].

In both of the cases, the matrix

$$\left[\begin{array}{c|c} A_i & B_i \\ \hline C_i & 0 \end{array}\right]$$

describes algebraic properties of the system S_i and is called *the matrix of the system* S_i. Thus, it is common use to identify the system S_i with the triple of matrices (A_i, B_i, C_i).

In this paper, we study special loop connections of the linear systems $S_1, \ldots, S_t$, obtained when the input of the system S_{j+1} is a linear function of the output of the system S_j, $j = 1, \ldots, t-1$, and when the input of the system S_1 is a linear function of the outputs of the systems $S_2, \ldots, S_t$. This is represented by the following equations:

$$\begin{aligned} u_1 &= F_1 y_2 + F_2 y_3 + \cdots + F_{t-1} y_t, \\ u_i &= K_{i-1} y_{i-1}, \quad i = 2, \ldots, t, \end{aligned}$$

where $F_j \in \mathbb{F}^{m_1 \times p_{j+1}}$ and $K_j \in \mathbb{F}^{m_{j+1} \times p_j}$, $j = 1, \ldots, t-1$. As a result of this connection, we obtain a new system S with the state $[x_1^T \quad \cdots \quad x_t^T]^T$ and the matrix

$$\begin{bmatrix} A_1 & B_1F_1C_2 & B_1F_2C_3 & \cdots & B_1F_{t-1}C_t \\ B_2K_1C_1 & A_2 & 0 & \ddots & 0 \\ 0 & B_3K_2C_2 & A_3 & 0 & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & B_tK_{t-1}C_{t-1} & A_t \end{bmatrix}. \tag{3}$$

As in [20] and [21], we only consider the linear systems S_j, with the properties $\operatorname{rank} B_j = n_j$, $\operatorname{rank} C_j = n_j$, $j = 1, \ldots, t$. Thus, the problem of determining the possible eigenvalues of (3) when matrices K_j and F_j, $j = 1, \ldots, t-1$, vary, is equivalent to the problem of determining the possible eigenvalues of the matrix

$$\begin{bmatrix} A_1 & Y_1 & Y_2 & \cdots & Y_{t-1} \\ X_1 & A_2 & 0 & \ddots & 0 \\ 0 & X_2 & A_3 & 0 & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & X_{t-1} & A_t \end{bmatrix}, \tag{4}$$

when $Y_j \in \mathbb{F}^{n_1 \times n_{j+1}}$, $X_j \in \mathbb{F}^{n_{j+1} \times n_j}$, $j = 1, \ldots, t-1$, vary.

In Theorem 2, we give a solution to the following problem:

Problem 1. Let $\mathbb{F}$ be an arbitrary field. Find necessary and sufficient conditions for the existence of matrices X_i, Y_i, $i = 1, \ldots, t-1$, over $\mathbb{F}$ such that matrix (4) has prescribed eigenvalues.

Similar problems have been studied from two different points of view: generic and non-generic.

The problems of describing the possible eigenvalues of a matrix when some of its entries are fixed and others vary, have been studied for a long time. London and Minc [15] proved that there always exists an $n \times n$ matrix over an arbitrary field $\mathbb{F}$ with prescribed eigenvalues and

$n-1$ entries (see also [8]). Furthermore, de Oliveira [17] proved that both eigenvalues and n entries can be prescribed, except in some special cases. Hershkowitz [13] extended this result to $2n-3$ entries. Moreover, Cravo and Silva [4] improved the last result by describing the possible eigenvalues of a $kp \times kp$ matrix, partitioned into $k \times k$ blocks $A_{ij} \in \mathbb{F}^{p\times p}$, when $2k-3$ of these blocks are fixed and others vary, see also [5]. For $k=2$ and if blocks A_{11} and A_{22} are prescribed and the others vary, the problem of determining the possible eigenvalues is solved by Silva [20]. The same author improved this result, when fixing the blocks A_{11}, A_{22} and A_{12} [21].

This approach can be extended to the study of the possible similarity class of a matrix when a submatrix is prescribed. For these and other non-generic linear algebra results describing the possible eigenvalues of a matrix see also [1,3,6,16,19,22–24]. Almost all of these problems have been studied over arbitrary fields. For other results and references, see also the book by Gohberg et al. [11].

Concerning the generic approach, one of the most general results is given by Helton et al. in [12]. They have showed that, given a complex linear subspace $\mathscr{L}$ of $\mathbb{C}^{n\times n}$, with dimension $\geqslant n$ and containing a matrix with nonzero trace, there exists a generic set of matrices in $\mathbb{C}^{n\times n}$ for which the characteristic map $\chi_A : \mathscr{L} \longrightarrow \mathbb{C}^n$, $L \longmapsto \det(\lambda I - A - L)$, $A \in \mathbb{C}^{n\times n}$, is generically surjective. For other resuts that use similar techniques see also [9,18] and their references. Also, there is a large literature on generic approach pole placement problems (e.g. [2,7]), where people were concerned with the minimal number of free parameters where arbitrary pole placement was still possible. By using this approach very sharp bounds on the number of free parameters can be given.

In this paper, we are using a non-generic approach and we give a complete solution of the Problem 1 for all possible matrices A_i, over arbitrary fields.

For $t=2$, the problem of finding necessary and sufficient conditions for the existence of matrices X_1 and Y_1 such that the matrix

$$\begin{bmatrix} A_1 & Y_1 \\ X_1 & A_2 \end{bmatrix}$$

has prescribed eigenvalues has been already solved in [20]. Thus, we are interested in solving Problem 1 when $t \geqslant 3$.

2. Notation

In this section, we introduce the notation that will be used throughout the paper, and give some basic linear algebra results.

Throughout the paper, we assume that all polynomials are monic. If f is a polynomial, $d(f)$ denotes its degree. If $f(\lambda) = \lambda^k - a_{k-1}\lambda^{k-1} - \cdots - a_1\lambda - a_0 \in \mathbb{F}[\lambda]$, $k > 0$, then $C(f)$ denotes the companion matrix

$$C(f) = \left[e_2^{(k)} \cdots e_k^{(k)} a\right]^{\mathrm{T}},$$

where $e_i^{(k)}$ is the ith column of the identity matrix I_k and

$$a = [a_0 \cdots a_{k-1}]^{\mathrm{T}}.$$

If $\psi_1 | \cdots | \psi_n$ are the invariant polynomials of a matrix $A \in \mathbb{F}^{n\times n}$, make convention that $\psi_i = 1$, for any $i \leqslant 0$, and $\psi_i = 0$, for any $i \geqslant n+1$. Let s be the number of nontrivial among $\psi_1 | \cdots | \psi_n$, i.e. the number of indices $i \in \{1, \ldots, n\}$ such that $\psi_i \neq 1$. The matrix A is similar to its normal form (see e.g. [10,14])

$$N(A) = C(\psi_{n-s+1}) \oplus \cdots \oplus C(\psi_n).$$

Also, if $A(\lambda) \in \mathbb{F}^{n\times m}$, $r = \operatorname{rank} A(\lambda)$, and $\psi_1|\cdots|\psi_r$ are the invariant factors of $A(\lambda)$, then $\psi_i = 1$, for any $i \leqslant 0$, and $\psi_i = 0$, for any $i \geqslant r+1$.

Definition 1. Let $A \in \mathbb{F}^{n\times n}$, $B \in \mathbb{F}^{n\times m}$. The pair (A, B) is said to be controllable if one of the following (equivalent) conditions is satisfied:

(1) $\min_{\lambda\in\overline{\mathbb{F}}} \operatorname{rank}[\lambda I - A \quad -B] = n$,
(2) all invariant factors of the matrix pencil $[\lambda I - A \quad -B]$ are trivial,
(3) $\operatorname{rank}[B \quad AB \quad A^2B \quad \cdots \quad A^{n-1}B] = n$.

By the characteristic polynomial of a polynomial matrix $D(\lambda) \in \mathbb{F}[\lambda]^{n\times m}$, we mean the product of its invariant factors.

3. Auxiliary results

The following theorem is the main result of [21], written in its transposed form, and will be used later in the proof of Theorem 2.

Theorem 1. *Let $\mathbb{F}$ be a field. Let $c_1, \ldots, c_{m+n} \in \mathbb{F}$, $A_{11} \in \mathbb{F}^{m\times m}$, $A_{21} \in \mathbb{F}^{n\times m}$, and $A_{22} \in \mathbb{F}^{n\times n}$. Let $f_1(\lambda)|\cdots|f_m(\lambda)$ be the invariant factors of*

$$\begin{bmatrix} \lambda I_m - A_{11} \\ -A_{21} \end{bmatrix}$$

and let $g_1(\lambda)|\cdots|g_n(\lambda)$ be the invariant factors of

$$\begin{bmatrix} \lambda I_n - A_{22} & -A_{21} \end{bmatrix}.$$

There exists $A_{12} \in \mathbb{F}^{n\times m}$ such that the matrix

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \tag{5}$$

has eigenvalues $c_1, \ldots, c_{m+n}$ if and only if the following conditions hold:

(a) $c_1 + \cdots + c_{n+m} = \operatorname{tr} A_{11} + \operatorname{tr} A_{22}$.
(b) $f_1(\lambda)\cdots f_m(\lambda) g_1(\lambda)\cdots g_n(\lambda) | (\lambda - c_1)\cdots(\lambda - c_{n+m})$.
(c) *One of the following conditions is satisfied:*
 (c$_1$) *For every $\nu \in \mathbb{F}$, $A_{21}A_{11} + A_{22}A_{21} \neq \nu A_{21}$.*
 (c$_2$) $A_{21}A_{11} + A_{22}A_{21} = \nu A_{21}$
 with $\nu \in \mathbb{F}$, and there exists a permutation
 $\pi : \{1, \ldots, m+n\} \to \{1, \ldots, m+n\}$ *such that*
 $c_{\pi(2i-1)} + c_{\pi(2i)} = \nu$
 for every $i = 1, \ldots, l$, where $l = \operatorname{rank} A_{21}$, and
 $c_{\pi(2l+1)}, \ldots, c_{\pi(m+n)}$
 are roots of $f_1(\lambda)\cdots f_m(\lambda) g_1(\lambda)\cdots g_n(\lambda)$.

The following lemma is used in the necessity part of the proof of the main result. Note that in the case $t = 1$ it reduces to the Sá-Thompson's result for polynomial matrices (see [19,22]).

Lemma 1. *Let $\mathbb{F}$ be a field. Let $A_i(\lambda) \in \mathbb{F}[\lambda]^{n_i \times n_i}$, $i = 1, \ldots, t$. Let $\alpha_1^j | \cdots | \alpha_{n_j}^j$ be the invariant factors of the matrices $A_j(\lambda)$, $j = 1, \ldots, t$. Let $\phi(\lambda) \in \mathbb{F}[\lambda]$ be a monic polynomial. If there exist matrices $X_i(\lambda) \in \mathbb{F}[\lambda]^{n_i \times n_{i-1}}$, $i = 1, \ldots, t$, such that the matrix*

$$\begin{bmatrix} X_1(\lambda) & A_1(\lambda) & & 0 \\ & \ddots & \ddots & \\ 0 & & X_t(\lambda) & A_t(\lambda) \end{bmatrix} \tag{6}$$

has $\phi(\lambda)$ as a characteristic polynomial, then the following condition is valid:

$$\prod_{i=1}^{n_1} \alpha_{i-w_1}^1 \prod_{i=1}^{n_2} \alpha_{i-w_2}^2 \prod_{i=1}^{n_3} \alpha_{i-w_3}^3 \cdots \prod_{i=1}^{n_t} \alpha_{i-w_t}^t | \phi(\lambda), \tag{7}$$

where $w_i = \min\{n_0, \ldots, n_i\}$, $i = 1, \ldots, t$.

Proof. Denote by M the set of indices $j \in \{1, \ldots, t\}$ such that $n_j < n_i$, for all $0 \leqslant i < j$. Let those indices be $s_1 + 1 \leqslant s_2 + 1 \leqslant \cdots \leqslant s_p + 1$, and let $s_0 = -1$ and $s_{p+1} = t$. Then for $s_i < j \leqslant s_{i+1}$, $i = 0, \ldots, p$, we have $w_j = n_{s_i+1}$.

Now, the condition (7) becomes

$$\begin{aligned} &\prod_{i=1}^{n_1} \alpha_{i-n_0}^1 \prod_{i=1}^{n_2} \alpha_{i-n_0}^2 \cdots \prod_{i=1}^{n_{s_1}} \alpha_{i-n_0}^{s_1} \times \\ &\prod_{i=1}^{n_{s_1+1}} \alpha_{i-n_{s_1+1}}^{s_1+1} \prod_{i=1}^{n_{s_1+2}} \alpha_{i-n_{s_1+1}}^{s_1+2} \cdots \prod_{i=1}^{n_{s_2}} \alpha_{i-n_{s_1+1}}^{s_2} \times \\ &\cdots \\ &\prod_{i=1}^{n_{s_p+1}} \alpha_{i-n_{s_p+1}}^{s_p+1} \prod_{i=1}^{n_{s_p+2}} \alpha_{i-n_{s_p+1}}^{s_p+2} \cdots \prod_{i=1}^{n_t} \alpha_{i-n_{s_p+1}}^{t} | \phi(\lambda). \end{aligned}$$

Further proof will go by induction on the number of elements in M, denoted by p.

Let $p = 0$. Thus, $n_0 \leqslant n_i$, $i = 1, \ldots, t$, and so $w_j = n_0$, $j = 1, \ldots, t$. In order to prove that

$$\prod_{j=1}^{t} \prod_{i=1}^{n_j} \alpha_{i-n_0}^j | \phi(\lambda), \tag{8}$$

we shall use the induction on t. Let $t = 1$. In this case the problem reduces to the following one:

If there exists a matrix $X_1(\lambda) \in \mathbb{F}[\lambda]^{n_1 \times n_0}$, such that the matrix

$$[A_1(\lambda) \quad X_1(\lambda)] \in \mathbb{F}[\lambda]^{n_1 \times (n_1 + n_0)} \tag{9}$$

has $\phi(\lambda)$ as a characteristic polynomial, then the following condition is valid

$$\prod_{i=1}^{n_1} \alpha_{i-n_0}^1 | \phi(\lambda).$$

Since the matrix (9) is equivalent to the matrix

$$[\mathrm{diag}(\bar{\alpha}_1^1, \dots, \bar{\alpha}_{n_1}^1) \quad 0] = [\overline{A}_1(\lambda) \quad 0], \tag{10}$$

where $\bar{\alpha}_1^1 | \cdots | \bar{\alpha}_{n_1}^1$ are the invariant factors of (9), by applying the Sá-Thompson result (see [19, 22]), we have

$$\bar{\alpha}_i^1 | \alpha_i^1 | \bar{\alpha}_{i+n_0}^1, \quad i = 1, \dots, n_1. \tag{11}$$

Thus,

$$\prod_{i=1}^{n_1} \alpha_{i-n_0}^1 \left| \prod_{i=1}^{n_1} \bar{\alpha}_i^1 = \phi(\lambda) \right.$$

as wanted.

Suppose now that the condition (8) is valid for $t-1$. Our aim is to prove that it will be valid for t. As in the previous case, matrix (9) is equivalent to (10). By applying these transformations on the matrix (6), it becomes equivalent to

$$\begin{bmatrix} 0 & \overline{A}_1(\lambda) & & & & 0 \\ X_2^1(\lambda) & X_2^2(\lambda) & A_2(\lambda) & & & \\ & & X_3(\lambda) & A_3(\lambda) & & \\ & & & \ddots & \ddots & \\ 0 & & & & X_t(\lambda) & A_t(\lambda) \end{bmatrix}, \tag{12}$$

where $X_2^1(\lambda) \in \mathbb{F}[\lambda]^{n_2 \times n_0}$ and $X_2^2(\lambda) \in \mathbb{F}[\lambda]^{n_2 \times n_1}$. Thus, the product of the characteristic polynomials of the matrices $\overline{A}_1(\lambda)$ and

$$\begin{bmatrix} X_2^1(\lambda) & A_2(\lambda) & & & 0 \\ & X_3(\lambda) & A_3(\lambda) & & \\ & & \ddots & \ddots & \\ 0 & & & X_t(\lambda) & A_t(\lambda) \end{bmatrix} \tag{13}$$

is equal to $\phi(\lambda)$. Now, it is enough to apply the induction hypothesis and the condition (11), and thus to obtain

$$\prod_{i=1}^{n_1} \alpha_{i-n_0}^1 \prod_{i=1}^{n_2} \alpha_{i-n_0}^2 \cdots \prod_{i=1}^{n_t} \alpha_{i-n_0}^t | \phi(\lambda),$$

as wanted.

Now suppose that the condition (7) is valid if the number of elements of the set M is equal to $p-1$. Our aim is to prove that it will be valid if the number of elements of the set M is equal to p. As in the previous case, we have that matrix (6) is equivalent to matrix (12). Thus, $\phi(\lambda)$ is equal to the product of the characteristic polynomials of the matrices $\overline{A}_1(\lambda)$ and (13). Now, put the matrix

$$[A_2(\lambda) \quad X_2^1(\lambda)] \in \mathbb{F}[\lambda]^{n_2 \times (n_2+n_0)} \tag{14}$$

into the equivalent form

$$[\mathrm{diag}(\bar{\alpha}_1^2, \dots, \bar{\alpha}_{n_2}^2) \quad 0] = [\bar{A}_2(\lambda) \quad 0],$$

where $\bar{\alpha}_1^2|\cdots|\bar{\alpha}_{n_2}^2$ are the invariant factors of matrix (14). Then we have

$$\alpha_i^2|\bar{\alpha}_i^2|\alpha_{i+n_0}^2, \quad i=1,\ldots,n_2.$$

Now, we can proceed so that in the ith step, $i=1,\ldots,s_1$, we obtain the matrix $\overline{A}_i(\lambda)=\operatorname{diag}(\bar{\alpha}_1^i,\ldots,\bar{\alpha}_{n_i}^i)$, where $\bar{\alpha}_1^i|\cdots|\bar{\alpha}_{n_i}^i$ are the invariant factors of the matrix

$$[A_i(\lambda) \quad X_i^1(\lambda)] \tag{15}$$

and such that matrix (15) is equivalent to the matrix

$$[\overline{A}_i(\lambda) \quad 0].$$

Thus, for every $i=1,\ldots,s_1$, we have

$$\alpha_j^i|\bar{\alpha}_j^i|\alpha_{j+n_0}^i, \quad j=1,\ldots,n_i. \tag{16}$$

After s_1 steps, we obtain that the product of the characteristic polynomials of the matrices $\bar{A}_i(\lambda)$, $i=1,\ldots,s_1$, and

$$\begin{bmatrix} X_{s_1+1}^1(\lambda) & A_{s_1+1}(\lambda) & & & 0 \\ & X_{s_1+2}(\lambda) & A_{s_1+2}(\lambda) & & \\ & & \ddots & \ddots & \\ 0 & & & X_t(\lambda) & A_t(\lambda) \end{bmatrix} \tag{17}$$

is equal to $\phi(\lambda)$. Here $X_{s_1+1}^1(\lambda)\in\mathbb{F}[\lambda]^{n_{s_1+1}\times n_0}$. Furthermore, since $n_{s_1+1}<n_0$, $X_{s_1+1}^1(\lambda)$ is equivalent to

$$\begin{bmatrix} 0 & \overline{X}_{s_1+1}^1(\lambda) \end{bmatrix}$$

for some $\overline{X}_{s_1+1}^1(\lambda)\in\mathbb{F}[\lambda]^{n_{s_1+1}\times n_{s_1+1}}$. Hence, the matrix (17) has the same invariant factors as the following one:

$$\begin{bmatrix} \overline{X}_{s_1+1}^1(\lambda) & A_{s_1+1}(\lambda) & & & 0 \\ & X_{s_1+2}(\lambda) & A_{s_1+2}(\lambda) & & \\ & & \ddots & \ddots & \\ 0 & & & X_t(\lambda) & A_t(\lambda) \end{bmatrix}. \tag{18}$$

Finally, the set $\overline{M}$ of all the indices $j\in\{s_1+2,\ldots,t\}$, such that $n_j<n_i$, for all $s_1+1\leqslant i<j$, has $p-1$ elements (and they are $s_2+1\leqslant\cdots\leqslant s_p+1$). So, we can apply the induction hypothesis on the matrix (18), and thus conclude

$$\prod_{i=1}^{n_1}\bar{\alpha}_i^1\cdots\prod_{i=1}^{n_{s_1}}\bar{\alpha}_i^{s_1}\prod_{i=1}^{n_{s_1+1}}\alpha_{i-n_{s_1+1}}^{s_1+1}\prod_{i=1}^{n_{s_1+2}}\alpha_{i-n_{s_1+1}}^{s_1+2}\cdots\prod_{i=1}^{n_{s_2}}\alpha_{i-n_{s_1+1}}^{s_2}\times$$
$$\cdots$$
$$\prod_{i=1}^{n_{s_p+1}}\alpha_{i-n_{s_p+1}}^{s_p+1}\prod_{i=1}^{n_{s_p+2}}\alpha_{i-n_{s_p+1}}^{s_p+2}\cdots\prod_{i=1}^{n_t}\alpha_{i-n_{s_p+1}}^{t}|\phi(\lambda).$$

Finally, from (16), follows

$$\prod_{i=1}^{n_j}\alpha_{i-n_0}^j\left|\prod_{i=1}^{n_j}\bar{\alpha}_i^j\right., \quad j=1,\ldots,s_1.$$

Thus, we have

$$\prod_{i=1}^{n_1} \alpha^1_{i-w_1} \prod_{i=1}^{n_2} \alpha^2_{i-w_2} \cdots \prod_{i=1}^{n_t} \alpha^t_{i-w_t} | \phi(\lambda)$$

as wanted. □

4. Main result

Theorem 2. *Let $\mathbb{F}$ be a field. Let $t \geqslant 3$. Let $A_i \in \mathbb{F}^{n_i \times n_i}$, $i = 1, \dots, t$. Let $w_i = \min\{n_1, \dots, n_i\}$, $i = 2, \dots, t$, $w_1 = w_2$. Let $m = \sum_{i=1}^t n_i$. Let $\alpha^i_1 | \cdots | \alpha^i_{n_i}$ be the invariant polynomials of the matrices A_i, $i = 1, \dots, t$. Let $c_1, \dots, c_m \in \mathbb{F}$. There exist matrices $X_i \in \mathbb{F}^{n_{i+1} \times n_i}$ and $Y_i \in \mathbb{F}^{n_1 \times n_{i+1}}$, $i = 1, \dots, t-1$, such that the matrix*

$$\left[\begin{array}{c|c|c|c} A_1 & Y_1 & \cdots & Y_{t-1} \\ \hline X_1 & A_2 & & 0 \\ & \ddots & \ddots & \\ 0 & & X_{t-1} & A_t \end{array}\right] \in \mathbb{F}^{m \times m} \tag{19}$$

has $c_1, \dots, c_m$ as eigenvalues if and only if:

(i) $\sum_{i=1}^t \operatorname{tr} A_i = \sum_{i=1}^m c_i$,
(ii) $\prod_{i=1}^{n_1} \alpha^1_{i-w_1} \prod_{i=1}^{n_2} \alpha^2_{i-w_2} \prod_{i=1}^{n_3} \alpha^3_{i-w_3} \cdots \prod_{i=1}^{n_t} \alpha^t_{i-w_t} | \phi(\lambda)$,
where $\phi(\lambda) = (\lambda - c_1)(\lambda - c_2) \cdots (\lambda - c_m)$.

Proof *Necessity*:

The necessity of the first condition follows trivially, and the necessity of the second one follows from Lemma 1. Indeed, by applying Theorem 1, we obtain that the product of the invariant factors of the matrices

$$\begin{bmatrix} \lambda I - A_1 \\ -X_1 \end{bmatrix} \in \mathbb{F}[\lambda]^{(n_1+n_2) \times n_1} \tag{20}$$

and

$$\begin{bmatrix} -X_1 & \lambda I - A_2 & & 0 \\ & \ddots & \ddots & \\ 0 & & -X_{t-1} & \lambda I - A_t \end{bmatrix} \tag{21}$$

divide $\phi(\lambda)$. Let $\beta_1 | \cdots | \beta_{n_1}$ be the invariant factors of matrix (20). By applying Sá-Thompson result we have

$$\prod_{i=1}^{n_1} \alpha^1_{i-n_2} \Bigg| \prod_{i=1}^{n_1} \beta_i. \tag{22}$$

Thus, by applying Lemma 1, we obtain the condition (ii), as wanted.

Sufficiency:

Suppose that the conditions (i) and (ii) are valid. Then the problem is equivalent to the following one:

Define matrices $X_i \in \mathbb{F}^{n_{i+1}\times n_i}$, $Y_i \in \mathbb{F}^{n_1\times n_{i+1}}$, $i = 1,\ldots,t-1$, such that the matrix

$$\left[\begin{array}{c|c|c|c} N(A_1) & Y_1 & \cdots & Y_{t-1} \\ \hline X_1 & N(A_2) & & 0 \\ & \ddots & \ddots & \\ 0 & & X_{t-1} & N(A_t) \end{array}\right] \tag{23}$$

has prescribed eigenvalues from the field $\mathbb{F}$.

Recall that $N(A_i)$ is the normal form for similarity of the matrix A_i, $i = 1,\ldots,t$.

Put the first $\sum_{i=1}^{n_1-w_1} d(\alpha_i^1)$ columns in X_1 to be zero. Furthermore, put the first $\sum_{i=1}^{n_1-w_1} d(\alpha_i^1)$ rows in the matrices Y_i, $i = 1,\ldots,t-1$, to be zero. Put the first $\sum_{i=1}^{n_j-w_j} d(\alpha_i^j)$ columns in the matrices Y_{j-1} and X_j, $j = 2,\ldots,t$, and the first $\sum_{i=1}^{n_j-w_j} d(\alpha_i^j)$ rows in the matrices X_{j-1}, $j = 2,\ldots,t$, to be zero.

Let $\overline{A}_i$ be a submatrix of $N(A_i)$ formed by its last $n_i - \sum_{j=1}^{n_i-w_i} d(\alpha_j^i)$ rows and columns, $i = 1,\ldots,t$. Then, $\alpha^j_{n_j-w_j+1}|\cdots|\alpha^j_{n_j}$ are the invariant polynomials of $\overline{A}_j$, $j = 1,\ldots,t$. Let $d_1,\ldots,d_x$ be elements from the field $\mathbb{F}$ such that

$$\sum_{i=1}^{t} \mathrm{tr}\overline{A}_i = \sum_{i=1}^{x} d_i,$$

$x = \sum_{j=1}^{t}\sum_{i=n_j-w_j+1}^{n_j} d(\alpha_i^j)$.

Let $a_1^i \geqslant \cdots \geqslant a_{k_i}^i$ be the degrees of the nontrivial invariant polynomials among $\alpha^i_{n_i-w_i+1}|\cdots|\alpha^i_{n_i}$, $i = 1,\ldots,t$. Thus, we have

$$w_i \geqslant k_i, \quad i = 1,\ldots,t \tag{24}$$

and $w_2 \geqslant \cdots \geqslant w_t$. Let $a^j = \sum_{i=1}^{k_j} a_i^j$, i.e. $\overline{A}_j \in \mathbb{F}^{a^j\times a^j}$, $j = 1,\ldots,t$. Now our problem reduces to the problem of defining matrices $\overline{X}_i \in \mathbb{F}^{a^{i+1}\times a^i}$ and $\overline{Y}_i \in \mathbb{F}^{a^1\times a^{i+1}}$, $i = 1,\ldots,t-1$, such that the matrix

$$\left[\begin{array}{c|c|c|c} \overline{A}_1 & \overline{Y}_1 & \cdots & \overline{Y}_{t-1} \\ \hline \overline{X}_1 & \overline{A}_2 & & 0 \\ & \ddots & \ddots & \\ 0 & & \overline{X}_{t-1} & \overline{A}_t \end{array}\right] \tag{25}$$

has $d_1,\ldots,d_x$ as eigenvalues.

In order to solve this problem we shall use the result from Theorem 1. Our aim is to define matrices $\overline{X}_i$, $i = 1,\ldots,t-1$, such that the matrices

$$\begin{bmatrix} \lambda I - \overline{A}_1 \\ -\overline{X}_1 \end{bmatrix}$$

and

$$\begin{bmatrix} -\overline{X}_1 & \lambda I - \overline{A}_2 & & 0 \\ & \ddots & \ddots & \\ 0 & & -\overline{X}_{t-1} & \lambda I - \overline{A}_t \end{bmatrix}, \tag{26}$$

have all invariant factors equal to 1.

First, we shall prove the following inequalities:

$$\min\{a^1, a^2\} \geqslant \max\{k_1, k_2\} \tag{27}$$
$$\min\{a^1, a^2, a^3\} \geqslant k_3 \tag{28}$$
$$\cdots$$
$$\min\{a^1, \ldots, a^t\} \geqslant k_t, \tag{29}$$

i.e.,

$$\min\{a^1, a^2\} \geqslant \max\{k_1, \ldots, k_t\} \tag{30}$$

and

$$a^i \geqslant \max\{k_{i+1}, \ldots, k_t\}, \quad i = 3, \ldots, t-1. \tag{31}$$

In order to prove (30), suppose that $\alpha^1_{n_1-w_2+1} = 1$. Then, $a^1 = \sum_{i=1}^{k_1} a^1_i = n_1$ and from (24), we have

$$a^1 \geqslant \max\{k_1, \ldots, k_t\}. \tag{32}$$

If, now, suppose that $\alpha^1_{n_1-w_2+1} \neq 1$, then $k_1 = w_2$ and $a^1 \geqslant w_2$. Again from (24), we obtain (32).

Analogously for a^2, we obtain

$$a^2 \geqslant \max\{k_1, \ldots, k_t\}. \tag{33}$$

Now, let $i \in \{3, \ldots, t\}$. If suppose that $\alpha^i_{n_i-w_i+1} = 1$, then $a^i = n_i$ and thus, from (24), we have

$$a^i \geqslant \max\{k_{i+1}, \ldots, k_t\}.$$

If now suppose that $\alpha^i_{n_i-w_i+1} \neq 1$, then $k_i = w_i$ and $a^i \geqslant w_i$. Again, from (24), we obtain (31), as wanted.

Finally, we can define $\overline{X}_1$ as follows: put $\min\{k_1, k_2\}$ units at the positions $\left(\sum_{i=1}^{j} a^2_{k_2-i+1}, \sum_{i=1}^{j-1} a^1_{k_1-i+1} + 1\right)$, $j = 1, \ldots, \min\{k_1, k_2\}$. Furthermore, put more $\max\{k_1, k_2\} - \min\{k_1, k_2\}$ units in $\overline{X}_1$ such that the rank of a such obtained matrix is equal to the $\max\{k_1, k_2\}$. More precisely, if $k_1 \geqslant k_2$, put $k_1 - k_2$ units in the columns $\sum_{i=1}^{j-1} a^1_{k_1-i+1} + 1$, $j = k_2 + 1, \ldots, k_1$ of $\overline{X}_1$. If $k_1 < k_2$, put $k_2 - k_1$ units in the rows $\sum_{i=1}^{j} a^2_{k_2-i+1}$, $j = k_1 + 1, \ldots, k_2$ of $\overline{X}_1$. Finally, put more $\min\{a^1, a^2\} - \max\{k_1, k_2\}$ units such that the rank of a such obtained matrix $\overline{X}_1$ is equal to the $\min\{a^1, a^2\}$.

Thus, the matrices

$$\begin{bmatrix} \lambda I - \overline{A}_1 \\ -\overline{X}_1 \end{bmatrix} \quad \text{and} \quad [\lambda I - \overline{A}_2 \quad -\overline{X}_1]$$

have all invariant factors equal to 1.

Now consider the following matrix

$$\begin{bmatrix} \overline{X}_1 & \overline{A}_2 & & 0 \\ & \ddots & \ddots & \\ 0 & & \overline{X}_{t-1} & \overline{A}_t \end{bmatrix}. \tag{34}$$

With the units in the matrix $\overline{X}_1$ as pivots, put zeros in the respective rows of $\overline{A}_2$ in the matrix (34). In this way, we have obtained $\min\{a^1, a^2\}$ zero columns in $\overline{A}_2$. Now, define $\overline{X}_2$ such that under the zero columns of the matrix $\overline{A}_2$, we put $\min\{a^1, a^2, a^3\}$ units such that k_3 of them are in the rows $\sum_{i=1}^{j} a^3_{k_3-i+1}$, $j = 1, \ldots, k_3$, respectively, and such that the matrix $\overline{X}_2$ has the rank equal to the $\min\{a^1, a^2, a^3\}$. All other entries in the matrix $\overline{X}_2$ put to be zeros. Now, we can proceed with the procedure, by defining the matrices $\overline{X}_i$'s with ranks equal to $\min\{a^1, \ldots, a^i\}$, $i = 1, \ldots, t-1$, such that matrix (26) has all invariant factors equal to 1.

Now, we are left to prove the existence of matrices $\overline{Y}_i$, $i = 1, \ldots, t-1$, such that the matrix (25) has $d_1, \ldots, d_x$ as eigenvalues. In order to apply the result from Theorem 1, we are left to prove that the condition (c) from the same theorem is valid.

Suppose that there exists $\nu \in \mathbb{F}$ such that

$$\begin{bmatrix} \overline{X}_1 \\ 0 \end{bmatrix} \overline{A}_1 + \begin{bmatrix} \overline{A}_2 & & & 0 \\ \overline{X}_2 & \overline{A}_3 & & \\ & \ddots & \ddots & \\ 0 & & \overline{X}_{t-1} & \overline{A}_t \end{bmatrix} \begin{bmatrix} \overline{X}_1 \\ 0 \end{bmatrix} = \begin{bmatrix} \nu \overline{X}_1 \\ 0 \end{bmatrix}. \tag{35}$$

Our aim is to prove that Eq. (35) is false for any $\nu \in \mathbb{F}$, i.e., that

$$\begin{bmatrix} \overline{X}_1 \\ 0 \end{bmatrix} \overline{A}_1 + \begin{bmatrix} \overline{A}_2 & & & 0 \\ \overline{X}_2 & \overline{A}_3 & & \\ & \ddots & \ddots & \\ 0 & & \overline{X}_{t-1} & \overline{A}_t \end{bmatrix} \begin{bmatrix} \overline{X}_1 \\ 0 \end{bmatrix} \neq \begin{bmatrix} \nu \overline{X}_1 \\ 0 \end{bmatrix} \tag{36}$$

for all $\nu \in \mathbb{F}$.

Eq. (35) is equivalent to the following ones

$$\overline{X}_1 \overline{A}_1 + \overline{A}_2 \overline{X}_1 = \nu \overline{X}_1 \tag{37}$$

and

$$\overline{X}_2 \overline{X}_1 = 0. \tag{38}$$

Since $\operatorname{rank} \overline{X}_1 = \min\{a^1, a^2\}$, there exist invertible matrices $P \in \mathbb{F}^{a^1 \times a^1}$ and $Q \in \mathbb{F}^{a^2 \times a^2}$ such that

$$\widetilde{X}_1 = Q \overline{X}_1 P = [I \quad 0] \in \mathbb{F}^{a^2 \times a^1}, \quad \text{if } a^2 \leqslant a^1, \tag{39}$$

or

$$\widetilde{X}_1 = Q \overline{X}_1 P = \begin{bmatrix} I \\ 0 \end{bmatrix} \in \mathbb{F}^{a^2 \times a^1}, \quad \text{if } a^1 \leqslant a^2. \tag{40}$$

Hence, the matrix (25) is similar to the following one

$$\left[\begin{array}{c|c|c|c|c} \widetilde{A}_1 & \widetilde{Y}_1 & \widetilde{Y}_2 & \cdots & \widetilde{Y}_{t-1} \\ \hline \widetilde{X}_1 & \widetilde{A}_2 & & & 0 \\ & \widetilde{X}_2 & \overline{A}_3 & & \\ & & \ddots & \ddots & \\ 0 & & & \overline{X}_{t-1} & \overline{A}_t \end{array}\right], \tag{41}$$

where $\widetilde{A}_1 = P^{-1} \overline{A}_1 P$, $\widetilde{Y}_1 = P^{-1} \overline{Y}_1 Q^{-1}$, $\widetilde{Y}_i = P^{-1} \overline{Y}_i$, $i = 2, \ldots, t-1$, $\widetilde{A}_2 = Q \overline{A}_2 Q^{-1}$ and $\widetilde{X}_2 = \overline{X}_2 Q^{-1}$.

First suppose that $a^2 \leqslant a^1$, i.e., that (39) is valid. Then, from (38), we have

$$\widetilde{X}_2\widetilde{X}_1 = 0,$$

i.e.,

$$\widetilde{X}_2[I \quad 0] = [\widetilde{X}_2 \quad 0] = 0.$$

Thus, $\widetilde{X}_2 = 0$, i.e. $\overline{X}_2 = 0$, which is a contradiction since the pair $(\overline{A}_3, \overline{X}_2)$ is controllable and $\dim \overline{A}_3 = a^3 > 0$.

Now, suppose $a^1 \leqslant a^2$. Then Eq. (40) is valid. Thus, from (37), we have

$$\widetilde{X}_1\widetilde{A}_1 + \widetilde{A}_2\widetilde{X}_1 = \nu\widetilde{X}_1, \tag{42}$$

i.e.,

$$\begin{bmatrix} I \\ 0 \end{bmatrix}\widetilde{A}_1 + \widetilde{A}_2\begin{bmatrix} I \\ 0 \end{bmatrix} = \begin{bmatrix} \nu I \\ 0 \end{bmatrix}. \tag{43}$$

Let $\widetilde{A}_2 = \begin{bmatrix} A_2^1 & A_2^2 \\ A_2^3 & A_2^4 \end{bmatrix}$, where $A_2^1 \in \mathbb{F}^{a^1 \times a^1}$. Then from (43), we obtain

$$\widetilde{A}_1 + A_2^1 = \nu I$$

and

$$A_2^3 = 0.$$

Since the pair (A_2^4, A_2^3) is controllable (this follows from the controllability of the pair $(\widetilde{A}_2, \widetilde{X}_1)$), and $A_2^3 = 0$, we have that $\widetilde{A}_2 = A_2^1$, i.e., that $a^1 = a^2$. So, $\widetilde{X}_1 = I$. Thus, we can apply the same arguments as in the case (39), and obtain that the condition (35) is not valid for any $\nu \in \mathbb{F}$. Finally, we can apply Theorem 1 and, thus, we finish the proof. □

5. Corollaries

Recall that a matrix $A \in \mathbb{C}^{m\times m}$ is said to be positively stable if and only if the real parts of all its eigenvalues are positive. Now as the direct consequence of Theorem 2, we have

Corollary 3. *Let $t \geqslant 3$. Let $A_i \in \mathbb{C}^{n_i\times n_i}$, $i = 1, \ldots, t$. Let $\alpha_1^i | \cdots | \alpha_{n_i}^i$ be the invariant polynomials of A_i, $i = 1, \ldots, t$. There exist matrices $X_i \in \mathbb{C}^{n_{i+1}\times n_i}$ and $Y_i \in \mathbb{C}^{n_1\times n_{i+1}}$, $i = 1, \ldots, t-1$, such that the matrix (19) is positively stable if and only if*

(i) $\mathrm{Re}(a_i) > 0, \quad i = 1, \ldots, k,$

(ii) $\mathrm{Re}\left(\sum_{i=1}^{t} \mathrm{tr} A_i\right) > \mathrm{Re}\left(\sum_{i=1}^{k} a_i\right),$

where $a_1, \ldots, a_k$ are all zeros (with multiplicities) of the polynomial

$$\prod_{i=1}^{n_1}\alpha_{i-w_1}^1 \prod_{i=1}^{n_2}\alpha_{i-w_2}^2 \prod_{i=1}^{n_3}\alpha_{i-w_3}^3 \cdots \prod_{i=1}^{n_t}\alpha_{i-w_t}^t,$$

$$w_i = \min\{n_1, \ldots, n_i\},\ i = 2, \ldots, t,\ w_1 = w_2.$$

Moreover, concerning the stability of discrete-time linear system, recall that (2) is stable if the modules of all the eigenvalues of the corresponding matrix of the system are less than 1. Now, from Theorem 2, we have

Corollary 4. *Let $t \geqslant 3$. Let $A_i \in \mathbb{C}^{n_i \times n_i}$, $i = 1, \dots, t$. Let $\alpha_1^i | \cdots | \alpha_{n_i}^i$ be the invariant polynomials of the matrices A_i, $i = 1, \dots, t$. There exist matrices $X_i \in \mathbb{C}^{n_{i+1} \times n_i}$ and $Y_i \in \mathbb{C}^{n_1 \times n_{i+1}}$, $i = 1, \dots, t-1$, such that the module of every eigenvalue of (19) is less than 1, if and only if*

(i) $|a_i| < 1, \quad i = 1, \dots, k$

(ii) $\left| \sum_{i=1}^t \operatorname{tr} A_i - \sum_{i=1}^k a_i \right| < m - k,$

where $a_1, \dots, a_k$ are all zeros (with multiplicities) of the polynomial

$$\prod_{i=1}^{n_1} \alpha_{i-w_1}^1 \prod_{i=1}^{n_2} \alpha_{i-w_2}^2 \prod_{i=1}^{n_3} \alpha_{i-w_3}^3 \cdots \prod_{i=1}^{n_t} \alpha_{i-w_t}^t,$$

$w_i = \min\{n_1, \dots, n_i\}$, $i = 2, \dots, t$, $w_1 = w_2$ *and* $m = \sum_{i=1}^t n_i$.

Corollary 5. *Let $\mathbb{F}$ be a field. Let $A_i \in \mathbb{F}^{s \times s}$, $i = 1, \dots, t$. Let $c_1, \dots, c_{ts} \in \mathbb{F}$. There exist matrices $X_i \in \mathbb{F}^{s \times s}$ and $Y_i \in \mathbb{F}^{s \times s}$, $i = 1, \dots, t-1$, such that the matrix*

$$\left[\begin{array}{c|c|c|c} A_1 & Y_1 & \cdots & Y_{t-1} \\ \hline X_1 & A_2 & & 0 \\ & \ddots & \ddots & \\ 0 & & X_{t-1} & A_t \end{array}\right] \tag{44}$$

has $c_1, \dots, c_{ts}$ as eigenvalues, if and only if:

$$\sum_{i=1}^t \operatorname{tr} A_i = \sum_{i=1}^{ts} c_i.$$

Proof. It is enough to prove that if $A_i \in \mathbb{F}^{s \times s}$, $i = 1, \dots, t$, then the conditions from the corollary and the ones from Theorem 2 are equivalent, i.e., that, in this particular case, the condition (ii) from Theorem 2 is trivially satisfied.

Indeed, by using the notation from the main result, if $n_1 = \cdots = n_t = s$, the condition (ii) from Theorem 2 becomes

$$\prod_{i=1}^s \alpha_{i-s}^1 \prod_{i=1}^s \alpha_{i-s}^2 \cdots \prod_{i=1}^s \alpha_{i-s}^t | \phi(\lambda), \tag{45}$$

where $\alpha_1^i | \cdots | \alpha_s^i$ are the invariant polynomials of A_i, $i = 1, \dots, t$. Thus, (45) is trivially satisfied, as wanted. □

From the proof of Theorem 2, we conclude that in the previous corollary we have defined the matrices X_i such that $X_i = I_s \in \mathbb{F}^{s \times s}$, for every $i = 1, \dots, t-1$. Thus, as a direct consequence of the previous result, we obtain the following theorem:

Corollary 6. *Let $\mathbb{F}$ be a field. Let $A_i \in \mathbb{F}^{s \times s}$, $i = 1, \dots, t$. Let $c_1, \dots, c_{ts} \in \mathbb{F}$. There exist matrices $Y_i \in \mathbb{F}^{s \times s}$, $i = 1, \dots, t-1$, such that the matrix*

$$\left[\begin{array}{c|c|c|c} A_1 & Y_1 & \cdots & Y_{t-1} \\ \hline I & A_2 & & 0 \\ & \ddots & \ddots & \\ 0 & & I & A_t \end{array}\right]$$

has $c_1, \ldots, c_{ts}$ *as eigenvalues if and only if*:

$$\sum_{i=1}^{t} \operatorname{tr} A_i = \sum_{i=1}^{ts} c_i.$$

Specially, if $A_i = 0$ *for* $i = 1, \ldots, t$, *then there exist matrices* $Y_i \in \mathbb{F}^{s \times s}$, $i = 1, \ldots, t-1$, *such that the matrix*

$$\left[\begin{array}{c|c|c|c} 0 & Y_1 & \cdots & Y_{t-1} \\ \hline I & 0 & & 0 \\ & \ddots & \ddots & \\ 0 & & I & 0 \end{array}\right]$$

has $c_1, \ldots, c_{ts}$ *as eigenvalues, if and only if*

$$\sum_{i=1}^{ts} c_i = 0.$$

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